EXAMPLE: MODELING THE PT326 PROCESS TRAINER

• The PT326 apparatus models common industrial situations in which temperature control is required in the presence of transport delay and transfer lag.

• Functionality:
  ○ Air drawn from the atmosphere by a centrifugal blower.
  ○ Air is heated as it passes over a heater grid.
  ○ Air is released into the atmosphere through a duct.

• Control Objectives:
  ○ Maintain the temperature of the air at a the output of the duct at a desired level.
  ○ Temperature control is achieved by varying the electrical power supplied to the heater grid.
  ○ The air temperature may be sensed by using a bead thermistor placed in the flow at any of the three positions along the duct.
  ○ The spatial separation between the thermistor and the heater coil introduces a transport delay into the system.

• The specific functionality features and settings are presented in the ENGG4420 Lab Manual.
**PROBLEM:** 1) Develop a dynamic model for the PT326 process trainer; 2) derive the transfer function of the process trainer model.

- Figure below shows the front panel of the PT326 apparatus. See the ENGG4420 Lab Manual for the description of the apparatus.

![Diagram of PT326 apparatus](image)

**SOLUTION:**

- The physical principle that governs the behaviour of the thermal process in PT326 apparatus is the balance of heat energy.
PT326 SYSTEM MODEL

Figure below shows a simplified graphical picture of the heat transfer process that takes place in the PT326 apparatus -- the volume V around the heater and the heat transfer rates are shown.

\[ qa = q + (q_i - q_o) - q_t \]  \hspace{1cm} (7)

• The rate at which heat accumulates in a fixed volume V enclosing the heater is:
  • \( q_a \) is the rate at which heat is supplied by the heater;
  • \( q_i \) is the rate at which heat is carried into the volume V by the coming air;
  • \( q_o \) is the rate at which heat is carried out of the volume V by the outgoing air; and
  • \( q_t \) is the heat lost from the volume V to the surroundings by radiation and conduction.
DERIVING THE MODEL EQUATION: for the PT326 apparatus we assume instantaneous heat exchange between the electric heater \( R_e \) and the air carried into the volume \( V \).

- The accumulation of heat in the volume \( V \) causes the temperature \( T \) of air in \( V \) to rise -- assuming a uniform temperature distribution in the volume \( V \), the rate of heat accumulation is also given based on Eq. (2) as:

\[
q_a = C \frac{dT}{dt}
\]  

Where \( C \) is the heat capacity of the air occupying the volume \( V \).

- Based on the assumption below:
  - Assume instant heat exchange between the electrical resistor \( R_e \) and the air flowing in volume \( V \)
  - Assume that all air coming into volume \( V \) leaves volume \( V \) instantly -- so, the incoming air doesn't stay in volume \( V \) and as a result, doesn't have any contribution to the net heat accumulated in \( V \) -- doesn't have time to exchange heat with the substance found in volume \( V \).

- We can conclude that the heat transfer due to Eq. (6) is zero so, \( q_i - q_o = 0 \); and the only net heat accumulated in volume \( V \) is due to heat transferred from heater \( R_e \) and the heat lost:
  - As a result, Eq. (5) becomes: \( q_a = q - q_t \);  

\[
(9)
\]
ASSUMING a small rise of temperature in volume V that is: \( \Delta T = T - T_a \); we can derive the equation model (SOLUTION to Problem 1) of the P326 apparatus as follows:

- Using Eq. (9) we can express:
  \[
  q_a + q_t = q \tag{10}
  \]

  \( q_a \) accumulated heat energy flow
  \( q_t \) heat energy flow
  \( q \) total heat in V
  \( q_a \) lost

  \( q_t \) from resistors

- \( q_a \) can be expressed by Eq. (8) and \( q_t \) by Eq. (1) and we get the following model equation for a small rise of temperature \( \Delta T \):
  \[
  C \frac{d\Delta T}{dt} + \frac{1}{R} \Delta T = q \tag{11}
  \]

  \( q_a \)
  \( q_t \)

- Note that \( q \) represents the heat flow generated by the heat resistors \( R_e \);
TRANSFER FUNCTION OF THE PROCESS TRAINER MODEL

Taking Laplace transform of Eq. (11) we get:

\[ L \cdot C \cdot \frac{d\Delta T}{dt} + \frac{1}{R} \Delta T_S = L \cdot \Delta q_S \]

**L Transform:** \(t\)-domain \(\rightarrow\) \(s\)-domain

\[ C \cdot s \cdot \Delta T(s) + \frac{1}{R} \Delta T(s) = g(s) \]

\[ \Delta T(s) \cdot \frac{R \cdot C \cdot s + 1}{R} = g(s) \]

Transfer function = \(\frac{\text{output}}{\text{input}}\)

\[ \frac{\Delta T(s)}{g(s)} = \frac{k_1}{\tau \cdot s + 1} \quad (12) \]

Where: \(k_1 = R\), and \(\tau = R \cdot C\) is the time constant.

• Note that here \(R\) and \(C\) relate to thermal resistance and heat capacity, respectively.
ASSUMING that the heat supply rate $q$ is proportional to the heater input voltage $V_i$ by a constant $k_2$, then Eq.(12) yields the transfer function between the temperature rise and the heater input voltage as:

$$\frac{\Delta T(s)}{V_i(s)} = \frac{k_1 \cdot k_2}{\tau \cdot s + 1}$$

(13)

$$\tau = \frac{R \cdot C}{\text{is the thermal time constant of the system}}$$

Where, $k_2$ is the proportionality constant between $q$ and $V_i$.

- In Eq. (13), $\Delta T$ represents the increase in temperature of the air in the volume $V$. The temperature sensor produces a voltage $V_o$ that is proportional to $\Delta T$, that is $V_o = k_3 \Delta T$.
- The sensor is physically located at a certain distance from the heat source and the sensor output responds to a temperature change with a pure time delay $\tau_d$.
- Then, the transfer function between the sensor's output voltage and the heater input voltage is:

$$\frac{V_o(s)}{V_i(s)} = \frac{k \cdot e^{-\tau_d \cdot s}}{\tau \cdot s + 1}$$

(14)
Where, \( k = k_1k_2k_3 \) and is the DC gain of the system.

The \( e^{(-\tau ds)} \) term in Eq. (14) arises due to fluid transport and is called a 'transport delay',

while term \( (\tau s + 1)^{-1} \) arises due to the heat transfer dynamics and is called 'transfer delay'.

Note the individual transfer block components in the diagram above.
The output of the temperature sensor $V_o$ and the heater input voltage $V_i$ are related by the first-order transfer function given by Eq. (14) for small temperature changes from a current state.

The transfer function in Eq. (14) is characterized by two parameters, namely:
- the DC gain $k$, and
- the time constant $\tau$.

Both of these parameters can be determined from the response of the temperature to a step increase in the heater input voltage from a state of thermal equilibrium.

Laplace transform of a unit step input function is $1/s$.

Laplace transform of the response of the temperature variation $\Delta V_o$ to an increase of 1V in the heater input is given in Eq. (15) and the inverse Laplace transform in Eq. (16) below.
LAPLACE TRANSFORM FOR THE STEP RESPONSE

\[ \frac{V_o(s)}{V_i(s)} = \frac{K \cdot e^{-\frac{\tau}{2}s}}{\tau \cdot s + 1} \quad (14) \]

\[ V_{\text{in}}(t) \quad \Downarrow \quad 1 \text{V} \quad \mathcal{L} \text{(step)} = \frac{1}{s} \]

\[ \Delta V_o(\tau) \quad \Delta V_o(0) \quad \Delta T \sim \Delta U_o \]

\[ V_o(s) = \frac{K \cdot e^{-\frac{\tau}{2}s}}{s \cdot (\tau \cdot s + 1)} \quad (15) \]

- The inverse Laplace transform of Eq. (15) is:

\[ V_o(t) = K \cdot (1 - e^{-\frac{t - \tau}{\tau}}) \quad (16) \]

\[ V_o(\tau + \tau_d) = K(1 - e^{-1}) = 0.63K \]

- Given \( V_i \) step inputs to the open loop PT326 apparatus for each blower opening we can record the voltage output (temperature) and determine \( \tau \) and \( \tau_d \).

- The experimental values for various blower openings are recorded in the lab manual.
The equation of motion of Newton's law is basic for obtaining a mathematical model for any mechanical system.

\[ F = m \cdot a \]  \hspace{1cm} (1)

Where,
- \( F \) = the sum of all forces applied to a body [N];
- \( a \) = inertial acceleration [m/sec\(^2\)];
- \( m \) = mass of the body [kg].

Application of Newton's law involves:
- a. Define convenient coordinates to account for the body's motion (position, velocity and acceleration);
- b. Determine the forces on the body using the free-body diagram;
- c. Write the equations of motion.

Newton's law applied to one-dimensional rotational system requires that the above equation be modified to:

\[ M = I \cdot \alpha \]  \hspace{1cm} (2)

Where,
- \( M \) = the sum of all external moments [Nm];
- \( I \) = the body's mass moment of inertia about its center of mass, [kg*m\(^2\)];
- \( \alpha \) = angular acceleration [rad/sec\(^2\)].
EXAMPLE: CRUISE CONTROL MODEL

- Write the equations of motion for the speed and forward motion of a car assuming that the engine develops a force $u$. Take the Laplace transform of the resulting differential equation and find the transfer function between the input $u$ (force) and output $v$ (speed).
- Use MATLAB to find the response of the velocity of the car for the case in which the input jumps from being $u = 0$ N at time $t = 0$ to a constant $u = 500$ N. Assume that the car mass is $m = 1,000$ kg and $b = 50$ N*sec/m.

SOLUTION

- We make the following assumptions:
  1. Rotational inertia of the wheels is negligible;
  2. The aerodynamic friction force opposing the motion of the car is proportional to speed $v$.

- The car can be approximated for modeling purposes by a free body diagram -- the coordinate of the car's position $x$ is the distance from the reference and is chosen so that positive is to the right.
In the case of the automotive cruise control the variable of the interest is the speed, \( v = \dot{x} \), and the equation of motion becomes:

\[
F = m \cdot a = (u - b \cdot \dot{x}) = m \cdot \ddot{x} \quad (1)
\]

• Eq. (2) is a first order differential equation in \( v \).
TIME RESPONSE USING MATLAB

- MATLAB can be used to plot the response using the transfer function of the system.
- The step function in MATLAB calculates the time response of a linear system to a unit step input.

\[
\begin{align*}
&\text{N} \\
&\text{O N} \\
&b = 0
\end{align*}
\]

- Because the system is linear, the output for this case can be multiplied by the magnitude of the input step to derive a step response of any magnitude. Equivalently, the numerator can be multiplied by the magnitude of the input step.
- SOLVE EQ. (2):

Assume a solution of the form: \( v = V_0 \cdot e^{st} \)

Given an input of the form: \( u = U_0 \cdot e^{st} \)

Then \( \dot{v} = s \cdot V_0 \cdot e^{st} \) goes into \( \text{Eq. (2)} \)

We get:

\[
(s + \frac{b}{m}) \cdot V_0 \cdot e^{st} = \frac{1}{m} \cdot U_0 \cdot e^{st}
\]  \( (3) \)
From Eq. (3) we set:

\[
\frac{v_0}{u_0} = \frac{1}{m}\frac{1}{s + \frac{b}{m}} \quad (4)
\]

Eq. (4) is usually written as:

\[
\frac{v(s)}{u(s)} = \frac{\frac{1}{m}}{s + \frac{b}{m}} \quad (5)
\]

- Eq. (5) -- is called "transfer function". In order to get the transfer function we substituted \( \frac{d}{dt} \) in Eq. (2) with \( s \) -- this is a general rule to obtain the transfer function from a differential equation.

**TIME RESPONSE** -- in order to obtain the time response using MATLAB we divide the transfer function into:

Numerator: \( \text{num} = \frac{1}{m} = 1/1000 \)

Denominator: \( \text{den} = [1 \ b/m] = [1 \ 50/1000] \)
MATLAB PROGRAM EXAMPLE

% program to find the time response for the cruise control system using the transfer function of Eq. (5)
>> num = 1/1000;   % b/m
>> den = [1 50/1000];   % s + b/m
>> sys = tf(num*500, den);
% step gives unit step response,
% so num*500 give u = 500 N
>> step(sys);     % plots the response
>> end

```
MATLAB PROGRAM EXAMPLE

% program to find the time response for the cruise control system using the transfer function of Eq. (5)
>> num = 1/1000;   % b/m
>> den = [1 50/1000];   % s + b/m
>> sys = tf(num*500, den);
% step gives unit step response,
% so num*500 give u = 500 N
>> step(sys);     % plots the response
>> end
```
EXAMPLE: A TWO MASS SYSTEM -- SUSPENSION MODEL

• Write the equations of motion for the automobile and wheel motion assuming one-dimensional vertical motion of one quarter car mass above the wheel.
  ○ Assume that the model is for a car with a mass of 1,580 kg, including the four wheels, which have a mass of 20 kg each; car deflection coefficient is: $k_s = 130,000 \text{ N/m}$; wheel deflection coefficient is: $k_w = 1,000,000 \text{ N/m}$; and $b = 9,800 \text{ N*sec/m}$.

TASKS OF AUTOMOTIVE SUSPENSION SYSTEMS

a. To provide good ride quality by isolating the car body from the road disturbances -- ride quality can be measured by the vertical acceleration of the passenger's locations.

b. To keep good road holding and handling on a rough and bumpy road, a winding road, and maneuvers of acceleration, lane change and braking -- road holding and handling can be represented by tire deflection.

c. To support the vehicle static weight -- measured by the suspension deflection (rattle space).
NOTES ON GRAVITATIONAL FORCES AND STABILITY

- Gravitational forces can always be omitted from vertical spring mass system if:
  - The position coordinates are defined from the equilibrium position that results when gravity is acting.
  - If the spring forces used in the analysis are actually the perturbation in spring forces from those forces acting at equilibrium.
- A stable system has always the same sign on similar variable of an equation of motion.

MODELING CONSIDRATIONS

- The suspension system can be approximated by the simplified free body diagram below (see next page).
- In the diagram, $x$ and $y$ represent the coordinates of masses $m_1$ and $m_2$ with respect to their equilibrium conditions -- the equilibrium positions are offset from the spring's unstretched positions because of the force of gravity.
- The shock absorber is modeled as a dashpot with friction constant $b$ -- the magnitude of the force from the shock absorber is assured to be proportional to the rate of the relative displacement of the 2 masses:

$$F_s = b \cdot (\dot{y} - \dot{x})$$
• By defining $x$ and $y$ to be the distance from the equilibrium position we can exclude the gravity forces.
• The force from the car suspension acts on both masses.
• For the lower spring, $k_w$ represents the tire compressibility for which there is insufficient damping (velocity dependent force) to include a dashpot -- the force from this spring is proportional to the distance the tire is compressed.
• By defining $x$ to be the distance from equilibrium, a force will result if either: 1) road surface has a bump ($r$ changes from its equilibrium value of zero), or 2) the wheel bounces ($x$ changes) -- the motion of the simplified car over a bumpy road will result in a variable $r(t)$. 
SUMMARY OF THE CAR SUSPENSION MODELING ELEMENTS

1. Car body and wheels -- modeled by their masses.
2. Shock absorber -- modeled by a dashpot with a friction constant b.
3. Springs -- modeled by Hooke's law ($F = -kx$) with $k$ being defined as the rate, spring constant or force constant of the spring. A spring stores potential elastic energy due to deformation (extension or contraction).
4. Tires -- modeled as springs
5. Road surface -- modeled as a time variable distance from equilibrium, $r(t)$.
6. The relative distance between car body and wheel ($y-x$) is responsible for generating the reactive forces in the spring and the shock absorber.
7. The relative distance between the wheel travel and road surface is responsible for generating the reactive forces in the tire.
EQUATIONS FOR THE MODEL

In the free body diagram:
- $m_2 =$ mass of the body, that is the sprung mass;
- $m_1 =$ mass of the wheel, that is the unsprung mass.

- The equations of motions can be written by applying Newton's law to each mass (See free body diagram).

\[ (1) \quad b(\ddot{y} - \dot{x}) + k_s (y - x) - k_w (x - \bar{r}) = m_1 \ddot{x} / m_1 \]

\[ (2) \quad -k_s (y - x) - b(\ddot{y} - \dot{x}) = m_2 \ddot{y} / m_2 \]

\[ \text{GOAL: Derive transfer function} \quad \frac{Y(s)}{R(s)} \]

\[ (1') \quad \ddot{x} + \frac{b}{m_1} (\ddot{x} - \dot{y}) + \frac{k_s}{m_1} (x - y) + \frac{k_w}{m_1} \cdot x = \frac{k_w}{m_1} \cdot \bar{r} \]

\[ (2') \quad \ddot{y} + \frac{b}{m_2} (\ddot{y} - \dot{x}) + \frac{k_s}{m_2} (y - x) = 0 \]

QUICK CHECK FOR SYSTEM STABILITY

Ex. In $E_s(1')$: $\dddot{x}$, $\dot{x}$, $x$ terms are all +

Likewise: $\ddot{y}$, $\dot{y}$, $y$ in $E_s(2')$ are all +

Etc. ...
TRANSFER FUNCTION: \( \frac{d}{dt} \rightarrow s \)

\[
\frac{Y(s)}{R(s)} = \frac{\frac{K_w \cdot b}{m_1 \cdot m_2}}{s + \frac{K_s}{b}} (s + \frac{K_s}{b})
\]

\[
= \frac{\frac{K_w \cdot b}{m_1 \cdot m_2}}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{K_s}{m_1} + \frac{K_s}{m_2} + \frac{K_w}{m_1}\right)s^2 + \frac{K_w b}{m_1} s + \frac{K_w K_s}{m_1 m_2}}
\]

**NUMERICAL EXAMPLE**

\[
m_2 = \frac{1580 \text{ kg} - 4 \times 20 \text{ kg}}{4} = 375 \text{ kg}
\]

\[
m_1 = 20 \text{ kg}
\]

\[
\frac{Y(s)}{R(s)} = \frac{1.31 \times 10^6 (s + 13.3)}{s^4 + 516.1 s^3 + 5.685 \times 10^4 s^2 + 1.307 \times 10^6 s + 1.733 \times 10^7}
\]
**HOMEWORK STUDY EXAMPLE:** ROTATIONAL MOTION -- SATELLITE ATTITUDE CONTROL MODEL

Satellites usually require attitude control so that antennas, sensors, and solar panels are properly oriented

- Antennas are usually pointed toward a particular location on earth, while solar panels need to be oriented toward the sun for maximum power generation.
- Note that attitude control is the orientation of a ship, spacecraft, or other flight vehicles either relative to celestial sphere or to a gravitating body influencing its flight path.

Controlling vehicle attitude requires sensors, actuators, and algorithms.

- Write the equations of motion for one axis of this system.

**SOLUTION**

**Modeling Considerations**

- Motion is allowed only about the axis perpendicular to the page;
- The angle $\theta$ describes the satellite orientation and must be measured with respect to an inertial reference;
- The control force comes from reaction jets that produce a moment of $F_c*d$ about the center;
- $M_D$ are small disturbance moments which arise primarily from solar pressure.

\[
(F_c \cdot d + M_D) = I \cdot \ddot{\theta} \quad \text{double integrator plant}
\]

Transfer Function: \[
\frac{\Theta(s)}{U(s)} = \frac{1}{I} \cdot \frac{1}{s^2} \quad ; \quad u = F_c \cdot d + M_D
\]
HOMEWORK STUDY EXAMPLE: ROTATIONAL MOTION -- PENDULUM

1. Write the equations of motion for a simple pendulum where all the mass is concentrated in the end point.
2. Use MATLAB to determine the time history of $\theta$ to a step input in $T_c$ of 1 Nm. Assume $l = 1m$, $m = 0.5$ kg, and $g = 9.81$ m/sec$^2$.

SOLUTION

$$M = l \cdot \alpha;$$

- Where $M$ is the external moment;
- $l$ is the moment of inertia and $\alpha$ is the angular acceleration.
- The moment of inertia about the pivot point is:
  - $l = m \cdot l^2$;
- The sum of moments about the pivot point contains 2 terms:
  - One from gravity
  - One from applied torque $T_c$.

$$T_c - m \cdot g \cdot l \cdot \sin \theta = I \cdot \ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l} \cdot \sin \theta = \frac{T_c}{m \cdot l^2}$$

(1)
LINIARIZATION OF THE PENDULUM MODEL

Assume motion is small \( \Rightarrow \sin \theta \approx \theta \)

As a result Eq (1) becomes:

\[
\ddot{\theta} + \frac{g}{e} \cdot \theta = \frac{T_c}{m \cdot e^2}
\]  

Transfer Function

\[
\frac{\Theta(s)}{T_c(s)} = \frac{1}{m \cdot e^2} \quad \frac{1}{s^2 + \frac{g}{e}}
\]

For MATLAB analysis we have:

\[
\text{num} = \frac{1}{m \cdot e^2} = \frac{1}{0.5 \cdot 1^2} = [27]
\]

\[
\text{den} = [1 \ 0 \ \frac{g}{e}] = [1 \ 0 \ 9.81]
\]
This is a classroom license for instructional use only. Research and commercial use is prohibited.

To get started, select MATLAB Help or Demos from the Help menu.

```matlab
% MATLAB PROGRAM FOR THE PENDULUM EXAMPLE
>> t = 0:0.001:10; % vector of time for outputs 0 to 10 at 0.02 increments
>> num = 2;
>> den = [1 0 9.81];
>> sys = tf(num, den); %define the system by its numerator and denominator

Transfer function:

\[ \frac{2}{s^2 + 9.81} \]

>> y = step(sys, t); % computes step responses at time given by vector t
>> plot(t, 57.3*y); % converts radians to degrees and plots step response
>>
```
ASSIGNMENTS

1. Develop the dynamic model for a flexible read/write disk drive [Reference: "Feedback Control of Dynamic Systems by G. F. Franklin et al."].
2. Study the 1/2 car and full car suspension models presented in the "ENGG4420 Lab Manual".