Finding a discrete equivalent to a given analog controller is equivalent to finding a recurrence equation for the samples of the control signal which will approximate the differential equations of the continuous controller.

ASSUME that we have the transfer function $D(s)$. We replace it with a discrete controller that will compute the next control signal to be sent to actuator by:

- Accepting samples of the controller input, $e(k\cdot T_s)$;
- Using past values of the control signal, $u(k\cdot T_s)$;
- Using present and past samples of the input, $e(k\cdot T_s)$.

EXAMPLE -- PID DISCRETE CONTROLLER

$$U(s) = \left( k_p + \frac{k_i}{s} + k_Ds \right) E(s),$$

(1)

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau) d\tau + k_D \dot{e}(t) = u_p + u_i + u_D,$$

(2)

where $T_s$ is the sample period.

For linear systems, the next control sample can be computed term by term and sum the terms (superposition).

THE PROPORTIONAL TERM $U_p$ is calculated below:

$$U_p(\kappa T_s + T_s) = k_p \cdot e(\kappa T_s + T_s);$$

(3)
THE INTEGRAL term can be computed by breaking the integral into two parts and approximating the second part, which is the integral over one sample period.

\[ U_I(\kappa \cdot T_s + T_s) = \]
\[ = k_I \cdot \int_{0}^{k \cdot T_s + T_s} e(\tau) \, d\tau \]
\[ = k_I \cdot \int_{0}^{k \cdot T_s} e(\tau) \, d\tau + \]
\[ + k_I \cdot \int_{k \cdot T_s}^{k \cdot T_s + T_s} e(\tau) \, d\tau = U_I(k \cdot T_s) + \frac{1}{2} \text{area under } e(\tau) \text{ over } T_s \]
\[ \approx U_I(k \cdot T_s) + k_I \cdot \frac{T_s}{2} \left( e(k \cdot T_s + T_s) + e(k \cdot T_s) \right) \tag{4} \]

- In Eq. (4) the area in question is approximated by that of a trapezoid formed by the base \( T_s \) and vertices \( e(kT_s+T_s) \) and \( e(kT_s) \), as shown by the dashed line of the figure above.
- Note that the area can also be approximated by the rectangle of amplitude \( e(kT_s) \) and width \( T_s \), shown by the solid blue in figure above:

\[ U_I(k \cdot T_s + T_s) = U_I(k \cdot T_s) + k_I \cdot T_s \cdot e(kT_s) \]
• THE DERIVATIVE TERM: the roles of \( u \) and \( e \) are reversed from integration. As a result, a simple approximation for this term can be derived using Eq. (4) as:

\[
k_0 \ e (\kappa \cdot \bar{t}_s + \bar{t}_s) = k_0 \ e (\kappa \cdot \bar{t}_s) + \frac{\bar{t}_s}{2} \left[ u_0 (\kappa \cdot \bar{t}_s + \bar{t}_s) + u_0 (\kappa \cdot \bar{t}_s) \right]
\]

We get:

\[
(5) \quad u_0 (\kappa \cdot \bar{t}_s + \bar{t}_s) = - u_0 (\kappa \cdot \bar{t}_s) + \frac{2 \cdot k_0}{\bar{t}_s} \left[ e (\kappa \cdot \bar{t}_s + \bar{t}_s) - e (\kappa \cdot \bar{t}_s) \right]
\]

SIMILAR to analog transfer functions, these relations are greatly simplified and generalized by the use of transform functions.

• In this course, we will introduce the discrete transform simply as a prediction operator \( z \), the same as we described the Laplace transform variable \( s \) as a differential operator.
USING THE DISCRETE TRANSFORM (Z-TRANSFORM)

- Prediction operator $z$ is a forward shift operator in the sense that if $U(z)$ is the transform of $u(k\cdot T_s)$, then $zU(z)$ will be the transform of $u(k\cdot T_s + T_s)$.

- USING THE SHIFT OPERATOR the integral term can be written as:

$$z U_I(z) = U_I(z) + K_I \cdot \frac{T_s}{T} \left[ \frac{z}{z-1} \cdot E(z) + E(z) \right];$$

$$U_I(z) = K_I \cdot \frac{T_s}{T} \cdot \frac{z+1}{z-1} \cdot E(z).$$

- From Eq. (5) the derivative term becomes the inverse:

$$U_D(z) = K_D \cdot \frac{2}{T_s} \cdot \frac{z-1}{z+1} \cdot E(z) \rightarrow \text{Inverse of}$$

- The complete PID controller is thus described by:

$$U(z) = \left( k_p + k_i \frac{T_s}{2} \frac{z+1}{z-1} + k_d \frac{2}{T_s} \frac{z-1}{z+1} \right) E(z), \quad (6)$$

For the PID controller example, the effect of the discrete approximation in the z-domain is obtained by replacing in the analog transfer function the operator $s$ with the composite operator:

$$s \Rightarrow \frac{2}{T_s} \frac{z-1}{z+1}$$

- This is the trapezoid rule of discrete equivalent, since we used the trapezoid approximations.
EXAMPLE -- EQUIVALENT DISCRETE CONTROLLER FOR SPEED CONTROL

* Plant transfer function

\[ G = \frac{Y}{U} = \frac{45}{(s+9)(s+5)} = \frac{9}{s+9} \cdot \frac{5}{s+5}, \]

* PI controller designated for the plant

\[ D(s) = \frac{U}{E} = 1.4 \frac{s+6}{s}. \]

The closed loop system has a rise time of about 0.2 sec and an overshoot of about 20%. Design a discrete equivalent of this controller, and compare the step responses and control signals of the two systems for:

a. \( T_s = 0.07 \), which is about 3 samples per rise time
b. \( T_s = 0.035 \), which is about 6 samples per rise time

SOLUTION TO EXAMPLE

Using the substitution given by the trapezoid rule the discrete equivalent for \( T_s = 0.07 \) is given by replacing \( s \) with \((2/0.07)(z-1)/(z+1)\) in \( D(s) \) as follows:
\[ \frac{U}{E} = D_q(z) = 1.4 \frac{\frac{2}{0.07} \cdot \frac{2-1}{2+1} + 6}{\frac{2}{0.07} \cdot \frac{2-1}{2+1}} = 1.4 \frac{1.21 \cdot \frac{2-1}{2-1}}{2-1} \]  

(1)

As a result, the equation for the controller is:

\[ U(z_c \cdot T_s + 1) = U(z_c) \cdot 2 \]

\[ U(k+1) = U(k) + 1.4 \left[ 1.21 \cdot e(k+1) - 0.79 \cdot e(k) \right] \]  

(2)

Note: In Eq(2) \( T_s \) is suppressed

Q: How did we set Eq(2)?

A: 

Care\( S \). For \( T_s = 0.035 \), the discrete transfer function is:

\[ \frac{U}{E} = D_1 = 1.4 \frac{1.105 \cdot z - 0.895}{z-1} \]  

(3)

As a result, the difference equation is:

\[ U(k+1) = U(k) + 1.4 \left[ 1.105 \cdot e(k+1) - 0.895 \cdot e(k) \right] \]  

(4)

Note: Eqs (2) & (4) give the control algorithms for the control task!!
SUPPLEMENTAL READING -- SIMULINK BLOCK DIAGRAM TO COMPARE CONTINUOUS, DISCRETE AND DIGITAL CONTROLLERS

- MATLAB provides a command that converts a continuous transfer function $D_c(s) = \frac{\text{numD}}{\text{denD}}$ represented as $\text{sysDa} = \text{tf(numD, denD)}$ to the discrete equivalent with sampling period $T_s$: $\text{sysDd} = \text{c2d(sysDa, Ts, 't')}$;
- After we get the discrete equivalent we can use this transfer function directly in the Simulink model.
- THE PLANT in the Simulink model will be represented as a continuous transfer function ($G = Y/U$) to better model the plant.
- The controllers will be expressed in the continuous and discrete forms ($D = U/E$) and their outputs are applied to the plant.
- An input (step) is applied to the input of the plant, which in our case is the reference input.
The output $u$ of the controllers that will show different output signals based on their type as: continuous or discrete.

1. The output of the plant that will also be showing different control performance.

2. From the designed Simulink system we can plot:
   - The discrete controller for $T_s = 0.07$ results in a substantial overshoot while for $T_s = 0.035$ the digital controller matches the performance of the analog controller fairly well.
   - For controllers with many poles and zeros we can use MATLAB to generate the discrete transfer function.
ASSIGNMENTS

1. Derive the equations for the DC motor example with the controller and develop the MATLAB program for plotting the step response for both cases: disturbance input W and reference input R.

2. [OPTIONAL] -- Implement the SIMULINK block diagram for EXAMPLE and plot the responses to a step input for the continuous and discrete controllers attached to the plant.
There are theoretical methods used to develop controllers that meet steady-state and transient specifications for both tracking input references and rejecting disturbances.

- These methods require that the designers have either a dynamic model of the process in the form of equations of motion or a detailed frequency response over a substantial range of frequencies. Either of these data can be difficult to obtain.

A more practical method is to specify satisfactory values for the controller settings based on estimates of the plant parameters that an operating engineer could make from experiments on the process itself - Ziegler-Nichols method.
ZIEGLER-NICHOLS TUNING OF PID CONTROLLER

- The step response of a large number of process control systems exhibit an S-shape process reaction curve.
- Ziegler-Nichols method approximates the step response of a real process system with the response of a first order system described by Eq. (1).
- The method is practical and can derive the controller settings based on estimates of the plant parameters obtained from experiments.

- The S-shape of the process reaction curve can be approximated by the step response of a first-order system with a time delay of $t_d$ seconds.
- The constants of the transfer function of Eq. (1) can be determined from the unit step response of the process.
  - NOTE: this is open loop response of the plant without the controller. Based on estimated R and L values we calculate the controller parameters using various estimation methods such as decay ratio or others (See Lab Manual).
- If a tangent is drawn at the inflection point of the reaction curve then the slope of the line is $R = \frac{A}{\tau}$ and the intersection point of the tangent line with the time axis identifies the time delay $L = t_d$.

\[
\frac{Y(s)}{U(s)} = \frac{Ae^{-st_d}}{\tau s + 1}, \quad (1)
\]
Controller parameters are chosen to result in a closed-loop step response transient with a decay ratio of approximately 0.25 after one period of oscillation.

The regulator parameters suggested by Ziegler and Nichols apply to the controller terms defined by:

\[ D_c(s) = k_p \left[ 1 + \frac{1}{T_i s} + T_d s \right], \]  

### PID PARAMETER VALUES

<table>
<thead>
<tr>
<th>Type of Controller</th>
<th>Optimum Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>( K_p = 1/R\cdot L )</td>
</tr>
<tr>
<td>I</td>
<td>( T_i = L/0.3 )</td>
</tr>
<tr>
<td>D</td>
<td>( T_d = 0.5 L )</td>
</tr>
<tr>
<td>PID</td>
<td>( K_p = 0.9/R\cdot L ) ( T_i = L/0.3 ) ( T_d = 0.5 L )</td>
</tr>
</tbody>
</table>
ZIEGLER-NICHOLS TUNING METHOD 2 -- ULTIMATE SENSITIVITY GAIN

- Method based on evaluating the amplitude and frequency of the oscillations of the system at the limit of stability rather than taking a step response.
- To use the method, the proportional gain in the system setup shown below is increased until the system becomes marginally stable and continuous oscillations just begin, with amplitude limited by the saturation of the actuator ($k_I$ and $k_D$ are set to 0, for this experiment).
- The corresponding gain is defined as $K_u$ and the period of oscillations as $P_u$ where $P_u$ should be measured when the amplitude of the oscillation is as small as possible.
- Experience has shown that the controller settings according to Zieglar-Nichols rules provide acceptable closed-loop response for many systems.
- The process operator will do final tuning of the controller iteratively on the actual process to yield satisfactory control.
BASED ON THE ULTIMATE SENSITIVITY METHOD the Ziegler-Nichols tuning parameters are:

\[
\begin{align*}
\rho & \rightarrow \quad K_p = 0.5 K_u \\
\rho I & \rightarrow \quad K_p = 0.45 K_u \\
& \quad T_I = \rho u / 1.2 \\
\rho I D & \rightarrow \quad K_p = 0.6 K_u \\
& \quad T_I = 0.5 \rho u \\
& \quad T_D = 0.125 \rho u
\end{align*}
\]

EXAMPLE -- TUNING OF A HEAT EXCHANGER USING THE QUARTER DECAY RATIO METHOD

Consider the a heat exchanger plant. The process reaction curve of this system is given in the next slide.

- Determine the PI regulator gains for the system using Ziegler-Nichols rules to achieve a quarter decay ratio.
- Plot the corresponding step responses.
Plant Model

\[ T_m(s) = \frac{K \cdot e^{-td}}{A_s(s) \cdot (\tau_1 \cdot s + 1)(\tau_2 + 1)} \]

\( T_m \) - measured temperature of water

\( A_s \) - area of valve

We measure: \( R = A/\tau \approx 1/90 \); \( L = 13 \text{ sec} \)

\( P \): \( K_P = 1/R_L = 90/13 = 6.92 \)

\( PI \): \( K_P = \frac{0.9}{R \cdot L} = 6.22 \); \( T_I = \frac{L}{0.3} = \frac{13}{0.3} = 43.3 \)

---

Fig. 2 -- step response using the parameters calculated above.

Fig. 3 -- step response for above parameters with \( k_p \) reduced by a factor of 2.
EXAMPLE -- TUNING A HEAT EXCHANGER BY USING THE OSCILLATORY BEHAVIOUR

- Proportional feedback was applied to the heat exchanger from this Example until the system showed non-decaying oscillations in response to a short pulse (impulse) input as shown in this figure:
  - The ultimate gain was $K_u = 15.3$ and the period was measured to be $P_u = 42$ sec.
- Determine the P and PI regulators based on the ultimate sensitivity method
  \[
  \text{The regulators are: } P: \quad K_P = 0.5 K_u = 7.65 \\
  \text{PI: } \quad K_P = 0.45 K_u = 6.825; \quad T_I = \frac{1}{1.2} P_u = 35
  \]

Fig. 2 -- step response of the closed-loop system using the above regulators.

Fig. 3 -- step response of the closed-loop system when $k_p$ is reduced by 50%.

Q: Why do we reduce $k_p$?
ASSIGNMENTS

1. Review all the homework, models and experimental examples presented so far, including the examples presented in the supplemental material.

2. Find the transfer function for the integrator circuit presented in figure below:

3. Study the model of the heat exchanger presented in the Feedback book. Develop the model in LabView and perform the tuning using the two Ziegler-Nichols methods. Choose your own data for the heat exchanger parameters.